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Exponential Stability of Uncertain Switched Systems with Multiple Non-differentiable Time-varying Delays

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Abstract

In this paper, the design of switching rule for exponential stability of a class of uncertain switched systems with delays is studied. The systems to be considered are linear and the state delays are time-varying. Based on Lyapunov functional and average dwell time approach, condition on the derivative of time-delay functions are not need to design switching rule for the exponential stability of switched systems with time-varying delays. The delay-dependent stability conditions are presented in terms of LMIs which can be solved easily by various available algorithms. The effectiveness of the proposed method is demonstrated by a simulation example.

Keywords: Exponential stability, Uncertain switched systems, Interval time-varying delays

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1 Introduction

During the past few decades, switched systems have been widely studied, [7], [11], [21-22], and reference cited therein. The switched systems are a special class of hybrid system which consist of a family of a differential or difference equations and a switching rule to indicate which subsystem will be activated at a specific interval of time. The motivation to study switched systems is mainly in twofold. Firstly, applications of switched systems may be found in engineering systems, such as automatic systems in air plane, car energy system, traffic system and machine industrial system, see [2-3]. Secondly, in certain situations, switching controller provides more desirable than a single controller, see [18]. The challenge to focus on the stability of switched systems lies in the fact that even if the individual systems are stable, the switched systems with some switching rules might be unstable, see [10]. Especially switched systems consisting of a family of stable and unstable subsystems can easily become unstable, see [1]. As the result, it is obviously important to study stability of switched systems.

Time-delay systems have been received considerable attention over the past decades. The main reason that the real processes in our world always involve time-delay systems, that is the present state depends on the past states which is the main sources of instability and less capable performance of the systems, see [4], [6], [17], [19], [21]. Therefore, it is significant to pay attention on time-delay systems.

Most of the results of switched systems are paid attention on analysis and design of the stability, see [10], [12-13], [14], [16]. In [8-9], [15], switched systems with time-varying delay were studied by using the common Lyapunov functional approach. The following restriction on the derivative of the delay is made, namely $\dot{d}(t) \leq D$ or $\dot{d}(t) \leq D < 1$. This restriction can lead to conservativeness, for example, it might not be used when the delay is non-differentiable or a fast time varying function. However, In [20], switched systems with multiple constant delays were established and obtained the delays dependent stability conditions. Most study mentioned above, some free weighting matrix variables are usually introduced. Based on Lyapunov stability theory, as well as Newton-Leibniz formula and average dwell time approach are used to obtain flexible and efficient stability condition of switched systems.

In this paper, we study the problem of exponential stability for a class of switched systems with switching multiple time-varying delays and consist of a family of stable and unstable subsystems. Compared with existing results in the literature, the novelty of our results is twofold. Firstly, the state delays are time-varying in which the restriction on the derivative of the time-delay functions are not required and the free weighting matrix variables are not need for the robust stability of the systems. Secondly, the obtained conditions for the exponential stability are delay-dependent and formulated in terms of the solution of standard LMIs which can be solved by various available algorithms, [5]. The paper is organized as follows. Section 2 presents notations, definitions and auxiliary propositions required for the proof of the main results. Switching design for the exponential stability of the system with illustrative examples are presented in Section 3 and 4, respectively. The paper ends with the conclusion followed by cited references.

2 Preliminary Notes / Materials and Methods

Throughout this paper, the following notations will be used.

$\mathbb{R}^n$ – the n dimensional Euclidean space;
$\mathbb{R}^{n\times n}$ – the set of all $n \times n$ real matrices;
$\mathbb{N}$ – the set of all positive integers;
$\|x\|$ – the Euclidean norm of vector $x \in \mathbb{R}^n$;
$\text{diag}\{\cdot\}$ – the block diagonal matrix;
I – the identity matrix;
A^T – the transpose of matrix A;
A^{-1} – the inverse of matrix A;
$\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ – $*$ represents the symmetric form of matrix, namely $* = B^T$;
$\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$ for any$A \in \mathbb{R}^{n\times m}$;
C_h – Space of continuous vector-valued function defined on$[-h, 0]$;
$x_t = x(t+\theta), -h \le \theta \le 0,$ where $x_t \in C_h$;
$\|x_t\|_{c_h} = \sup\limits_{-h\le\theta\le 0} \|x(t+\theta)\|$;
$\lambda_{\max}(A) = \max\{\text{Re}(\lambda) : \lambda \text{ is eigenvalue of } A\}$;
$\lambda_{\min}(A) = \min\{\text{Re}(\lambda) : \lambda \text{ is eigenvalue of } A\}$;
$M = \{1, 2, \ldots, N\}; M_u = \{1, 2, \ldots, r\}; M_s = \{r+1, r+2, \ldots, N\}$.

Consider the following uncertain switched system with time varying delay

$$\begin{aligned}
\dot{x}(t) &= (A_{\sigma(t)} + \Delta A_{\sigma(t)}(t))x(t) + (D_{1,\sigma(t)} + \Delta D_{1,\sigma(t)}(t))x(t - h_{\sigma(t)}(t)) \\
&\quad +(D_{2,\sigma(t)} + \Delta D_{2,\sigma(t)}(t))x(t - \tau_{\sigma(t)}(t)), t > 0, \\
x(t) &= \phi(t), t \in [-h, 0], \qquad (1)
\end{aligned}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector. $A_i, D_{1,i}, D_{2,i}$, $i \in M$ are known constant matrices, $\Delta A_i, \Delta D_{1,i}, \Delta D_{2,i}$ are uncertainty matrices which are of the form

$$\Delta A_i = E_{1,i}F_{1,i}(t)H_{1,i}, \Delta D_{1,i} = E_{2,i}F_{2,i}(t)H_{2,i}, \Delta D_{2,i} = E_{3,i}F_{3,i}(t)H_{3,i}, F_{j,i}^T(t)F_{j,i}(t) \le I,$$

$j = 1, 2, 3$, where $F_{1,i}(t), F_{2,i}(t), F_{3,i}(t)$ are unknown matrices, I is the identity matrix of appropriate dimension; $h_i(t)$ and $\tau_i(t)$ are the delay functions for an i^{th} subsystem which satisfy the following conditions

$$0 \le h_1 \le h_{1,i} \le h_i(t) \le h_{2,i} \le h_2, \quad (2)$$

$$0 \le \tau_1 \le \tau_{1,i} \le \tau_i(t) \le \tau_{2,i} \le \tau_2. \quad (3)$$

Let $h = max\{h_2, \tau_2\}$ and let $\phi(t)$ be an initial condition. $\sigma(t) : R^+ \cup \{0\} \to M$, $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, $\sigma(t)$ is called the switching signal, we have the switching sequence $\{x_{t_0}; (i_0, t_0), ...|i_k \in M, k = 0, 1, 2, ...\}$, which means that when $t \in [t_k, t_{k+1})$, the i_k^{th} subsystem is activated.
Let M_u and M_s be respectively the set of indices of the unstable and stable modes.

Definition 2.1. *The systems (2.1) is said to be exponentially stable under $\sigma(t)$, if the solution $x(t)$ to systems (2.1) satisfies*

$$\|x(t)\| \le \kappa \|x_{t_0}\|_{c_h} e^{\lambda(t-t_0)}, \forall t \ge t_0,$$

for constant $\kappa \ge 1$ and $\lambda > 0$.

Lemma 2.2. *(Schur Complement Lemma) Given constant symmetric matrices $Q,\ S$
and $R \in \mathbb{R}^{n\times n}$ where $R > 0,\ Q = Q^T$ and $R = R^T$ we have*

$$\begin{bmatrix} Q & S \\ S^T & -R \end{bmatrix} < 0 \Leftrightarrow \ Q + SR^{-1}S^T < 0.$$

Lemma 2.3. *([8]) Given $\varepsilon > 0$ and matrices D, E and F with $F^TF \le I$, we have*

$$DEF + E^TF^TD^T \le \varepsilon DD^T + \varepsilon^{-1}E^TE.$$

Lemma 2.4. *([8]) Given a positive definite matrix $P \in \mathbb{R}^{n\times n}$, any symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, then*

$$\lambda_{\min}(P^{-1}Q)x^TPx \le x^TQx \le \lambda_{\max}(P^{-1}Q)x^TPx.$$

Lemma 2.5. *([8]) For any positive semidefinite matrix $M \in \mathbb{R}^{n\times n}$, a scalar $\gamma > 0$ and a vector function $\omega : [0, \gamma] \to \mathbb{R}^{n\times n}$ such that the integrals concerned are well-defined, we have*

$$(\int_0^\gamma \omega(s)ds)^TM(\int_0^\gamma \omega(s)ds) \le \gamma \int_0^\gamma \omega^T(s)M\omega(s)ds.$$

Lemma 2.6. *(Cauchy inequality) For any symmetric positive definite matrix $N \in M^{m\times m}$ and $x, y \in \mathbb{R}^n$ we have*

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

For simplicity of later presentation, we give the following notations.
$N(t)$ denotes the switching number of the whole systems during (t_0, t);
$l(t)$ denotes the switching number of the unstable subsystems during (t_0, t);
$\Sigma_{s11,i} = 0.5P_i[A_i + \alpha I]^T + 0.5[A_i + \alpha I]P_i - (e^{-\alpha h_{1,i}} + e^{-\alpha h_{2,i}})R_{1,i} + 2Q_{1,i}$;
$\Sigma_{s12,i} = 0.5P_{1,i}A_i^T$;
$\Sigma_{s13,i} = e^{-\alpha h_{1,i}}R_{1,i}$;
$\Sigma_{s14,i} = e^{-\alpha h_{2,i}}R_{1,i}$;
$\Sigma_{s15,i} = D_{1,i}P_{1,i}$;
$\Sigma_{s22,i} = (h_{1,i}^2 + h_{2,i}^2)R_{1,i} + (h_{2,i} - h_{1,i})^2 U_{1,i} - P_{1,i}$;
$\Sigma_{s25,i} = D_{1,i}P_{1,i}$;
$\Sigma_{s33,i} = -e^{-\alpha h_{1,i}}Q_{1,i} - e^{-\alpha h_{1,i}}R_{1,i} - e^{-\alpha h_{2,i}}U_{1,i}$;
$\Sigma_{s35,i} = e^{-\alpha h_{2,i}}U_{1,i}$;
$\Sigma_{s44,i} = -e^{-\alpha h_{1,i}}Q_{1,i} - e^{-\alpha h_{1,i}}R_{1,i} - e^{-\alpha h_{2,i}}U_{1,i}$;
$\Sigma_{s45,i} = -e^{-\alpha h_{2,i}}U_{1,i}$;
$\Sigma_{s55,i} = -2e^{-\alpha h_{2,i}}U_{1,i}$;
$\Gamma_{s11,i} = P_{2,i}[A_i + \alpha I]^T + [A_i + \alpha I]P_{2,i} - (e^{-\alpha \tau_{1,i}} + e^{-\alpha \tau_{2,i}})R_{2,i} + 2Q_{2,i}$;
$\Gamma_{s12,i} = P_{2,i}A_i^T$;
$\Gamma_{s13,i} = e^{-\alpha \tau_{1,i}}R_{2,i}$;
$\Gamma_{s14,i} = e^{-\alpha \tau_{2,i}}R_{2,i}$;
$\Gamma_{s15,i} = D_{2,i}P_{2,i}$;
$\Gamma_{s22,i} = (\tau_{1,i}^2 + \tau_{2,i}^2)R_{2,i} + (\tau_{2,i} - \tau_{1,i})^2 U_{2,i} - 2P_{2,i}$;
$\Gamma_{s25,i} = D_{2,i}P_{2,i}$;
$\Gamma_{s33,i} = -e^{-\alpha \tau_{1,i}}Q_{2,i} - e^{-\alpha \tau_{1,i}}R_{2,i} - e^{-\alpha \tau_{2,i}}U_{2,i}$;
$\Gamma_{s35,i} = e^{-\alpha \tau_{2,i}}U_{2,i}$;
$\Gamma_{s44,i} = -e^{-\alpha \tau_{1,i}}Q_{2,i} - e^{\alpha \tau_{1,i}}R_{2,i} - e^{\alpha \tau_{2,i}}U_{2,i}$;
$\Gamma_{s45,i} = -e^{-\alpha \tau_{2,i}}U_{2,i}$;
$\Gamma_{s55,i} = -2e^{-\alpha \tau_{2,i}}U_{2,i}$;
$\Sigma_{u11,i} = 0.5P_i[A_i - \beta I]^T + 0.5[A_i - \beta I]P_i - (e^{\beta h_{1,i}} + e^{\beta h_{2,i}})R_{1,i} + 2Q_{1,i}$;
$\Sigma_{u12,i} = 0.5P_{1,i}A_i^T$;
$\Sigma_{u13,i} = e^{\beta h_{1,i}}R_{1,i}$;
$\Sigma_{u14,i} = e^{\beta h_{2,i}}R_{1,i}$;
$\Sigma_{u15,i} = D_{1,i}P_{1,i}$;
$\Sigma_{u22,i} = (h_{1,i}^2 + h_{2,i}^2)R_{1,i} + (h_{2,i} - h_{1,i})^2 U_{1,i} - P_{1,i}$;
$\Sigma_{u25,i} = D_{1,i}P_{1,i}$;
$\Sigma_{u33,i} = -e^{\beta h_{1,i}}Q_{1,i} - e^{\beta h_{1,i}}R_{1,i} - e^{\beta h_{2,i}}U_{1,i}$;
$\Sigma_{u35,i} = e^{\beta h_{2,i}}U_{1,i}$;
$\Sigma_{u44,i} = -e^{\beta h_{1,i}}Q_{1,i} - e^{\beta h_{1,i}}R_{1,i} - e^{\beta h_{2,i}}U_{1,i}$;
$\Sigma_{u45,i} = -e^{\beta h_{2,i}}U_{1,i}$;

$\Sigma_{u55,i} = -2e^{\beta h_{2,i}} U_{1,i};$
$\Gamma_{u11,i} = P_{2,i}[A_i - \beta I]^T + [A_i - \beta I]P_{2,i} - (e^{\beta\tau_{1,i}} + e^{\beta\tau_{2,i}})R_{2,i} + 2Q_{2,i};$
$\Gamma_{u12,i} = P_{2,i}A_i^T;$
$\Gamma_{u13,i} = e^{\beta\tau_{1,i}} R_{2,i};$
$\Gamma_{u14,i} = e^{\beta\tau_{2,i}} R_{2,i};$
$\Gamma_{u15,i} = D_{2,i}P_{2,i};$
$\Gamma_{u22,i} = (\tau_{1,i}^2 + \tau_{2,i}^2)R_{2,i} + (\tau_{2,i} - \tau_{1,i})^2 U_{2,i} - 2P_{2,i};$
$\Gamma_{u25,i} = D_{2,i}P_{2,i};$
$\Gamma_{u33,i} = -e^{\beta\tau_{1,i}} Q_{2,i} - e^{\beta\tau_{1,i}} R_{2,i} - e^{\beta\tau_{2,i}} U_{2,i};$
$\Gamma_{u35,i} = e^{\beta\tau_{2,i}} U_{2,i};$
$\Gamma_{u44,i} = -e^{\beta\tau_{1,i}} Q_{2,i} - e^{\alpha\tau_1} R_{2,i} - e^{\beta\tau_{2,i}} U_{2,i};$
$\Gamma_{u45,i} = -e^{\beta\tau_{2,i}} U_{2,i};$
$\Gamma_{u55,i} = -2e^{\beta\tau_{2,i}} U_{2,i};$
$\lambda_1 = \min_{i\in M}\{\lambda_{\min}\ (P_{1,i}^{-1})\} + \min_{i\in M}\{\lambda_{\min}\ (P_{2,i}^{-1})\};$

$$\begin{aligned}\lambda_2 = {} & \max_{i\in M}\{\lambda_{\max}\ (P_{1,i}^{-1})\} + 2h_{2,i}\max_{i\in M}\{\lambda_{\max}\ (P_{1,i}^{-1}Q_{1,i}P_{1,i}^{-1})\} \\ & +(h_{2,i} - h_{1,i})^2 \max_{i\in M}\{\lambda_{\max}\ (P_{1,i}^{-1}U_{1,i}P_{1,i}^{-1})\} + \max_{i\in M}\{\lambda_{\max}\ (P_{2,i}^{-1})\} \\ & +2\tau_{2,i}\max_{i\in M}\{\lambda_{\max}\ (P_{2,i}^{-1}Q_{2,i}P_{2,i}^{-1})\} + (\tau_{2,i} - \tau_{1,i})^2 \max_{i\in M}\{\lambda_{\max}\ (P_{2,i}^{-1}U_{2,i}P_{2,i}^{-1})\};\end{aligned}$$

$$\Omega_{si} = \begin{bmatrix} \Sigma_{s11,i} & \Sigma_{s12,i} & \Sigma_{s13,i} & \Sigma_{s14,i} & \Sigma_{s15,i} \\ * & \Sigma_{s22,i} & 0 & 0 & \Sigma_{s25,i} \\ * & * & \Sigma_{s33,i} & 0 & \Sigma_{s35,i} \\ * & * & * & \Sigma_{s44,i} & \Sigma_{s45,i} \\ * & * & * & * & \Sigma_{s55,i} \end{bmatrix};$$

$$\Phi_{si} = \begin{bmatrix} \Gamma_{s11,i} & \Gamma_{s12,i} & \Gamma_{s13,i} & \Gamma_{s14,i} & \Gamma_{s15,i} \\ * & \Gamma_{s22,i} & 0 & 0 & \Gamma_{s25,i} \\ * & * & \Gamma_{33,i} & 0 & \Gamma_{s35,i} \\ * & * & * & \Gamma_{s44,i} & \Gamma_{s45,i} \\ * & * & * & * & \Gamma_{s55,i} \end{bmatrix};$$

$$\Omega_{ui} = \begin{bmatrix} \Sigma_{u11,i} & \Sigma_{u12,i} & \Sigma_{u13,i} & \Sigma_{u14,i} & \Sigma_{u15,i} \\ * & \Sigma_{u22,i} & 0 & 0 & \Sigma_{u25,i} \\ * & * & \Sigma_{u33,i} & 0 & \Sigma_{u35,i} \\ * & * & * & \Sigma_{u44,i} & \Sigma_{u45,i} \\ * & * & * & * & \Sigma_{u55,i} \end{bmatrix};$$

$$\Phi_{ui} = \begin{bmatrix} \Gamma_{u11,i} & \Gamma_{u12,i} & \Gamma_{u13,i} & \Gamma_{u14,i} & \Gamma_{u15,i} \\ * & \Gamma_{u22,i} & 0 & 0 & \Gamma_{u25,i} \\ * & * & \Gamma_{u33,i} & 0 & \Gamma_{u35,i} \\ * & * & * & \Gamma_{u44,i} & \Gamma_{u45,i} \\ * & * & * & * & \Gamma_{u55,i} \end{bmatrix}.$$

3 Results and Discussion

3.1 Exponential stability for nominal switched systems with multiple interval time-varying delays.

The nominal switched systems is given by

$$\begin{aligned} &\dot{x}(t) = A_{\sigma(t)}x(t) + D_{1,\sigma(t)}x(t - h_{\sigma(t)}(t)) + D_{2,\sigma(t)}x(t - \tau_{\sigma(t)}(t)), t > 0 \\ &x(t) = \phi(t), \in [-h, 0], \end{aligned} \tag{4}$$

We now state the main result on sufficient condition for exponential stability of switched systems (3.1).

Theorem 3.1. *Let $\alpha > 0$ and $\beta > 0$ be the decay rate of the stable modes and the growth rate of the unstable modes, respectively. If there exists symmetric positive definite matrices $P_i, Q_{1,i}, Q_{2,i}, R_{1,i}, R_{2,i}, U_{1,i}, U_{2,i}$ such that the following conditions hold:*

$A1-i$. For $i \in M_s$

$$\Omega_{s1,i} = \Omega_{si} - [0 \quad 0 \quad 0 \quad -I \quad I]^T \times e^{-\alpha h_{2,i}} U_{1,i} [0 \quad 0 \quad 0 \quad -I \quad I] < 0, \tag{5}$$

$$\Omega_{s2,i} = \Omega_{si} - [0 \quad 0 \quad I \quad 0 \quad -I]^T \times e^{-\alpha h_{2,i}} U_{1,i} [0 \quad 0 \quad I \quad 0 \quad -I] < 0, \tag{6}$$

$$\Phi_{s1,i} = \Phi_{si} - [0 \quad 0 \quad 0 \quad -I \quad I]^T \times e^{-\alpha \tau_{2,i}} U_{2,i} [0 \quad 0 \quad 0 \quad -I \quad I] < 0, \tag{7}$$

$$\Phi_{s2,i} = \Phi_{si} - [0 \quad 0 \quad I \quad 0 \quad -I]^T \times e^{-\alpha \tau_{2,i}} U_{2,i} [0 \quad 0 \quad I \quad 0 \quad -I] < 0, \tag{8}$$

$A1-ii$. For $i \in M_u$

$$\Omega_{u1,i} = \Omega_{ui} - [0 \quad 0 \quad 0 \quad -I \quad I]^T \times e^{\beta h_{2,i}} U_{1,i} [0 \quad 0 \quad 0 \quad -I \quad I] < 0, \tag{9}$$

$$\Omega_{u2,i} = \Omega_{ui} - [0 \quad 0 \quad I \quad 0 \quad -I]^T \times e^{\beta h_{2,i}} U_{1,i} [0 \quad 0 \quad I \quad 0 \quad -I] < 0, \tag{10}$$

$$\Phi_{u1,i} = \Phi_{ui} - [0 \quad 0 \quad 0 \quad -I \quad I]^T \times e^{\beta \tau_{2,i}} U_{2,i} [0 \quad 0 \quad 0 \quad -I \quad I] < 0, \tag{11}$$

$$\Phi_{u2,i} = \Phi_{ui} - [0 \quad 0 \quad I \quad 0 \quad -I]^T \times e^{\beta \tau_{2,i}} U_{2,i} [0 \quad 0 \quad I \quad 0 \quad -I] < 0, \tag{12}$$

$A2$. For $\mu \geq 1$,

$$P_i \leq \mu P_j, Q_{r,i} \leq \mu Q_{r,j}, R_{r,i} \leq \mu R_{r,j}, U_{r,i} \leq \mu U_{r,j}, r = 1, 2, \forall i, j \in M. \tag{13}$$

A3. Let $T^+(t_0,t), T^-(t_0,t)$ denote the total activation times of the unstable and stable modes over (t_0,t), respectively. Assume that, for any t_0, the switching law guarantees that

$$\inf_{t\geq 0}\frac{T^-(t_0,t)}{T^+(t_0,t)} \geq \frac{\beta+\alpha^*}{\alpha+\alpha^*} \tag{14}$$

where $\alpha^* \in (0,\alpha)$; furthermore, there exists $0<\nu<\alpha^*$ such that
(i) for $i \in M_s$

$$\ln\mu + \alpha h - \nu(t_k - t_{k-1}) \leq 0, k = r+1, r+2, ..., N. \tag{15}$$

(ii) for $i \in M_u$

$$\ln\mu - \nu(t_k - t_{k-1}) \leq 0, k = 1,2,...,r. \tag{16}$$

Then switched systems (3.1) is exponentially stable. Moreover, the solution $x(t)$ of the switched systems satisfy

$$\|x(t)\| \leq \sqrt{\tfrac{\lambda_1}{\lambda_2}}\|x_{t_0}\|_{c_h} e^{-\frac{1}{2}(\alpha^*-\nu)(t-t_0)},\ t\geq t_0.$$

Proof. Let $Y_i = P_i^{-1}, y(t) = Y_i x(t)$. For $i \in M_s$, we choose the Lyapunov-Krasovskii functional candidates as

$$V_i(x_t) = V_{1,i}(x_t) + ... + V_{11,i}(x_t), \tag{17}$$

where,
$V_{1,i}(x_t) = x^T(t)Y_i x(t)$,
$V_{2,i}(x_t) = \int_{t-h_{1,i}}^{t} x^T(s)e^{\alpha(s-t)}Y_iQ_{1,i}Y_ix(s)ds$,
$V_{3,i}(x_t) = \int_{t-h_{2,i}}^{t} x^T(s)e^{\alpha(s-t)}Y_iQ_{1,i}Y_ix(s)ds$,
$V_{4,i}(x_t) = h_{1,i}\int_{-h_{1,i}}^{0}\int_{t+s}^{t} \dot{x}^T(\theta)e^{\alpha(\theta-t)}Y_iR_{1,i}Y_i\dot{x}(\theta)d\theta ds$,
$V_{5,i}(x_t) = h_{2,i}\int_{-h_{2,i}}^{0}\int_{t+s}^{t} \dot{x}^T(\theta)e^{\alpha(\theta-t)}Y_iR_{1,i}Y_i\dot{x}(\theta)d\theta ds$,
$V_{6,i}(x_t) = (h_{2,i}-h_{1,i})\int_{-h_{2,i}}^{-h_{1,i}}\int_{t+s}^{t} \dot{x}^T(\theta)e^{\alpha(\theta-t)}Y_iU_{1,i}Y_i\dot{x}(\theta)d\theta ds$,
$V_{7,i}(x_t) = \int_{t-\tau_{1,i}}^{t} x^T(s)e^{\alpha(s-t)}Y_iQ_{2,i}Y_ix(s)ds$,

$$V_{8,i}(x_t) = \int_{t-\tau_{2,i}}^{t} x^T(s)e^{\alpha(s-t)}Y_iQ_{2,i}Y_ix(s)ds,$$
$$V_{9,i}(x_t) = \tau_{1,i}\int_{-\tau_{1,i}}^{0}\int_{t+s}^{t} \dot{x}^T(\theta)e^{\alpha(\theta-t)}Y_iR_{2,i}Y_i\dot{x}(\theta)d\theta ds,$$
$$V_{10,i}(x_t) = \tau_{2,i}\int_{-\tau_{2,i}}^{0}\int_{t+s}^{t} \dot{x}^T(\theta)e^{\alpha(\theta-t)}Y_iR_{2,i}Y_i\dot{x}(\theta)d\theta ds,$$
$$V_{11,i}(x_t) = (\tau_{2,i}-\tau_{1,i})\int_{-\tau_{2,i}}^{-\tau_{1,i}}\int_{t+s}^{t} \dot{x}^T(\theta)e^{\alpha(\theta-t)}Y_iU_{2,i}Y_i\dot{x}(\theta)d\theta ds,$$

It's easy to see that

$$\lambda_1\|x(t)\|^2 \le V_i(x_t) \le \lambda_2\|x_t\|^2. \tag{18}$$

Taking the derivative of $V_i(x_t)$ with respect to t along any trajectory of solution of (3.1) yields

$$\begin{aligned}
\dot{V}_{1,i}(x_t) &= 2x^T(t)Y_i\dot{x}(t),\\
&= y^T(t)[P_iA_i^T + A_iP_i]y(t) + 2y^T(t)D_{1,i}P_iy(t-h_i(t))\\
&\quad +2y^T(t)D_{2,i}P_iy(t-\tau_i(t)), && (19)\\
\dot{V}_{2,i}(x_t) &= y^T(t)Q_{1,i}y(t) - e^{-\alpha h_{1,i}}y^T(t-h_{1,i})Q_{1,i}y(t-h_1) - \alpha V_{2,i}, && (20)\\
\dot{V}_{3,i}(x_t) &= y^T(t)Q_{1,i}y(t) - e^{-\alpha h_{2,i}}y^T(t-h_{2,i})Q_{1,i}y(t-h_2) - \alpha V_{3,i}, && (21)\\
\dot{V}_{4,i}(x_t) &\le h_{1,i}^2\dot{y}^T(t)R_{1,i}\dot{y}(t) - h_1e^{-\alpha h_{1,i}}\int_{t-h_{1,i}}^{t}\dot{y}^T(s)R_{1,i}\dot{y}(s)ds - \alpha V_{4,i}, && (22)\\
\dot{V}_{5,i}(x_t) &\le h_{2,i}^2\dot{y}^T(t)R_{1,i}\dot{y}(t) - h_{2,i}e^{-\alpha h_{2,i}}\int_{t-h_{2,i}}^{t}\dot{y}^T(s)R_{1,i}\dot{y}(s)ds - \alpha V_{5,i}, && (23)\\
\dot{V}_{6,i}(x_t) &\le (h_{2,i}-h_{1,i})^2\dot{y}^T(t)U_{1,i}\dot{y}(t) - (h_{2,i}-h_{1,i})e^{-\alpha h_{2,i}}\int_{t-h_{2,i}}^{t-h_{1,i}}\dot{y}^T(s)U_{1,i}\dot{y}(s)ds\\
&\quad -\alpha V_{6,i}, && (24)\\
\dot{V}_{7,i}(x_t) &= y^T(t)Q_{2,i}y(t) - e^{-\alpha\tau_{1,i}}y^T(t-\tau_{1,i})Q_{2,i}y(t-\tau_{1,i}) - \alpha V_{7,i}, && (25)
\end{aligned}$$

$$\begin{aligned}
\dot{V}_{8,i}(x_t) &= y^T(t)Q_{2,i}y(t) - e^{-\alpha\tau_{2,i}}y^T(t-\tau_{2,i})Q_{2,i}y(t-\tau_{2,i}) - \alpha V_{8,i}, && (26)\\
\dot{V}_{9,i}(x_t) &\le \tau_{1,i}^2\dot{y}^T(t)R_{2,i}\dot{y}(t) - \tau_{1,i}e^{-\alpha\tau_{1,i}}\int_{t-\tau_{1,i}}^{t}\dot{y}^T(s)R_{2,i}\dot{y}(s)ds - \alpha V_{9,i}, && (27)\\
\dot{V}_{10,i}(x_t) &\le \tau_{2,i}^2\dot{y}^T(t)R_{2,i}\dot{y}(t) - \tau_{2,i}e^{-\alpha\tau_{2,i}}\int_{t-\tau_{2,i}}^{t}\dot{y}^T(s)R_{2,i}\dot{y}(s)ds - \alpha V_{10,i}, && (28)\\
\dot{V}_{11,i}(x_t) &\le (\tau_{2,i}-\tau_{1,i})^2\dot{y}^T(t)U_{2,i}\dot{y}(t) - (\tau_{2,i}-\tau_{1,i})e^{-\alpha\tau_{2,i}}\int_{t-\tau_{2,i}}^{t-\tau_{1,i}}\dot{y}^T(s)U_{2,i}\dot{y}(s)ds\\
&\quad -\alpha V_{11,i}. && (29)
\end{aligned}$$

Then by applying Lemma 2.5 and Leibniz-Newton formular, we obtain

$$
\begin{aligned}
-h_{1,i}\int_{t-h_{1,i}}^{t}\dot{y}^T(s)R_{1,i}\dot{y}(s)ds \ &\le\ y^T(t)(-R_{1,i})y(t)+2y^T(t)R_{1,i}y(t-h_{1,i})\\
&\quad +y^T(t-h_{1,i})(-R_{1,i})y(t-h_{1,i}), \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
-h_{2,i}\int_{t-h_{1,i}}^{t}\dot{y}^T(s)R_{1,i}\dot{y}(s)ds \ &\le\ y^T(t)(-R_{1,i})y(t)+2y^T(t)R_{1,i}y(t-h_{2,i})\\
&\quad +y^T(t-h_{2,i})(-R_{1,i})y(t-h_{2,i}), \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
-\tau_{1,i}\int_{t-\tau_{1,i}}^{t}\dot{y}^T(s)R_{2,i}\dot{y}(s)ds \ &\le\ y^T(t)(-R_{2,i})y(t)+2y^T(t)R_{2,i}y(t-\tau_{1,i})\\
&\quad +y^T(t-\tau_{1,i})(-R_{2,i})y(t-\tau_{1,i}), \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
-\tau_{2,i}\int_{t-\tau_{1,i}}^{t}\dot{y}^T(s)R_{2,i}\dot{y}(s)ds \ &\le\ y^T(t)(-R_{2,i})y(t)+2y^T(t)R_{2,i}y(t-\tau_{2})\\
&\quad +y^T(t-\tau_{2,i})(-R_{2,i})y(t-\tau_{2,i}). \qquad (33)
\end{aligned}
$$

Note that

$$
\begin{aligned}
-(h_{2,i}-h_{1,i})\int_{t-h_{2,i}}^{t-h_{1,i}}\dot{y}^T(s)U_{1,i}\dot{y}(s)ds \ &=\ -(h_{2,i}-h_{1,i})\int_{t-h_{2,i}}^{t-h_i(t)}\dot{y}^T(s)U_{1,i}\dot{y}(s)ds\\
&\quad -(h_{2,i}-h_{1,i})\int_{t-h_i(t)}^{t-h_{1,i}}\dot{y}^T(s)U_{1,i}\dot{y}(s)ds\\
&=\ -(h_{2,i}-h_i(t))\int_{t-h_{2,i}}^{t-h_i(t)}\dot{y}^T(s)U_{1,i}\dot{y}(s)ds\\
&\quad -(h_i(t)-h_{1,i})\int_{t-h_{2,i}}^{t-h_i(t)}\dot{y}^T(s)U_{1,i}\dot{y}(s)ds\\
&\quad -(h_i(t)-h_{1,i})\int_{t-h_i(t)}^{t-h_{1,i}}\dot{y}^T(s)U_{1,i}\dot{y}(s)ds\\
&\quad -(h_{2,i}-h_i(t))\int_{t-h_i(t)}^{t-h_{1,i}}\dot{y}^T(s)U_{1,i}\dot{y}(s)ds.
\end{aligned}
$$

Similarly, we obtain that

$$
\begin{aligned}
-(\tau_{2,i}-\tau_{1,i})\int_{t-\tau_{2,i}}^{t-\tau_{1,i}} \dot{y}^T(s)U_{2,i}\dot{y}(s)ds \;=\;& -(\tau_{2,i}-\tau_i(t))\int_{t-\tau_{2,i}}^{t-\tau_i(t)} \dot{y}^T(s)U_{2,i}\dot{y}(s)ds \\
& -(\tau_i(t)-\tau_{1,i})\int_{t-\tau_{2,i}}^{t-\tau_i(t)} \dot{y}^T(s)U_{2,i}\dot{y}(s)ds \\
& -(\tau_i(t)-\tau_{1,i})\int_{t-\tau_i(t)}^{t-\tau_{1,i}} \dot{y}^T(s)U_{2,i}\dot{y}(s)ds \\
& -(\tau_{2,i}-\tau_i(t))\int_{t-\tau_i(t)}^{t-\tau_{1,i}} \dot{y}^T(s)U_{2,i}\dot{y}(s)ds.
\end{aligned}
$$

Using Lemma 2.5 yields

$$
\begin{aligned}
& -(h_{2,i}-h_i(t))\int_{t-h_{2,i}}^{t-h_i(t)} \dot{y}^T(s)(U_{1,i})\dot{y}(s)ds \\
& \le -[y(t-h_i(t))-y(t-h_{2,i})]^T U_{1,i}[y(t-h_i(t))-y(t-h_{2,i})], \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
& -(h_i(t)-h_{1,i})\int_{t-h_i(t)}^{t-h_{1,i}} \dot{y}^T(s)(U_{1,i})\dot{y}(s)ds \\
& \le -[y(t-h_{1,i})-y(t-h_i(t))]^T U_{1,i}[y(t-h_{1,i})-y(t-h_i(t))], \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
& -(\tau_{2,i}-\tau_i(t))\int_{t-\tau_{2,i}}^{t-\tau_i(t)} \dot{y}^T(s)(U_{2,i})\dot{y}(s)ds \\
& \le -[y(t-\tau_i(t))-y(t-\tau_{2,i})]^T U_{2,i}[y(t-\tau_i(t))-y(t-\tau_{2,i})], \qquad (36)
\end{aligned}
$$

and

$$
\begin{aligned}
& -(\tau_i(t)-\tau_{1,i})\int_{t-\tau_i(t)}^{t-\tau_{1,i}} \dot{y}^T(s)(U_{2,i})\dot{y}(s)ds \\
& \le -[y(t-\tau_{1,i})-y(t-\tau_i(t))]^T U_{2,i}[y(t-\tau_{1,i})-y(t-\tau_i(t))]. \qquad (37)
\end{aligned}
$$

Let $\beta_h = \frac{h_{2,i}-h_i(t)}{h_{2,i}-h_{1,i}} \le 1$, $\beta_\tau = \frac{\tau_{2,i}-\tau_i(t)}{\tau_{2,i}-\tau_{1,i}} \le 1$. Then

$$
\begin{aligned}
-(h_{2,i}-h_i(t))\int_{t-h_i(t)}^{t-h_{1,i}} \dot{y}^T(s)U_i\dot{y}(s)ds \;=\;& -\beta_h\int_{t-h_i(t)}^{t-h_{1,i}} (h_{2,i}-h_{1,i})\dot{y}^T(s)U_{1,i}\dot{y}(s)ds \\
\le\;& -\beta_h\int_{t-h_i(t)}^{t-h_{1,i}} (h_i(t)-h_{1,i})\dot{y}^T(s)U_{1,i}\dot{y}(s)ds \\
\le\;& -\beta_h[y(t-h_{1,i})-y(t-h_i(t))]^T U_{1,i} \\
& \times[y(t-h_{1,i})-y(t-h_i(t))], \qquad (38)
\end{aligned}
$$

$$\begin{aligned}
-(h_i(t)-h_{1,i})\int_{t-h_{2,i}}^{t-h_i(t)}\dot{y}^T(s)U_{1,i}\dot{y}(s)ds &= -(1-\beta_h)\int_{t-h_{2,i}}^{t-h_i(t)}(h_{2,i}-h_{1,i})\dot{y}^T(s)U_{1,i}\dot{y}(s)ds\\
&\leq -(1-\beta_h)\int_{t-h_{2,i}}^{t-h_i(t)}(h_{2,i}-h_i(t))\dot{y}^T(s)U_{1,i}\dot{y}(s)ds\\
&\leq -(1-\beta_h)[y(t-h_i(t))-y(t-h_{2,i})]^TU_{1,i}\\
&\quad\times[y(t-h_i(t))-y(t-h_{2,i})], \qquad (39)
\end{aligned}$$

$$\begin{aligned}
-(\tau_{2,i}-\tau_i(t))\int_{t-\tau_i(t)}^{t-\tau_{1,i}}\dot{y}^T(s)U_{2,i}\dot{y}(s)ds &= -\beta_\tau\int_{t-\tau_i(t)}^{t-\tau_1}(\tau_{2,i}-\tau_{1,i})\dot{y}^T(s)U_{2,i}\dot{y}(s)ds\\
&\leq -\beta_\tau\int_{t-\tau_i(t)}^{t-\tau_{1,i}}(\tau_i(t)-\tau_{1,i})\dot{y}^T(s)U_{2,i}\dot{y}(s)ds\\
&\leq -\beta_\tau[y(t-\tau_{1,i})-y(t-\tau_i(t))]^TU_{2,i}\\
&\quad\times[y(t-\tau_{1,i})-y(t-\tau_i(t))], \qquad (40)
\end{aligned}$$

and

$$\begin{aligned}
-(\tau_i(t)-\tau_{1,i})\int_{t-\tau_{2,i}}^{t-\tau_i(t)}\dot{y}^T(s)U_{2,i}\dot{y}(s)ds &= -(1-\beta_\tau)\int_{t-\tau_{2,i}}^{t-\tau_i(t)}(\tau_{2,i}-\tau_{1,i})\dot{y}^T(s)U_{2,i}\dot{y}(s)ds\\
&\leq -(1-\beta_\tau)\int_{t-\tau_{2,i}}^{t-\tau_i(t)}(\tau_{2,i}-\tau_i(t))\dot{y}^T(s)U_{2,i}\dot{y}(s)ds\\
&\leq -(1-\beta_\tau)[y(t-\tau_i(t))-y(t-\tau_{2,i})]^TU_{2,i}\\
&\quad\times[y(t-\tau_i(t))-y(t-\tau_{2,i})]. \qquad (41)
\end{aligned}$$

Therefore from (3.26)-(3.38), we have

$$\begin{aligned}
&-(h_{2,i}-h_{1,i})\int_{t-h_{2,i}}^{t-h_{1,i}}\dot{y}^T(s)U_{1,i}\dot{y}(s)ds\\
&\leq [y(t-h_i(t))-y(t-h_{2,i})]^T(-U_{1,i})[y(t-h_i(t))-y(t-h_{2,i})]\\
&+[y(t-h_i(t))-y(t-h_{2,i})]^T(-U_{1,i})[y(t-h_i(t))-y(t-h_{2,i})]\\
&-\beta_h[y(t-h_{1,i})-y(t-h_i(t))]^T(U_{1,i})[y(t-h_{1,i})-y(t-h_i(t))]\\
&-(1-\beta_h)[y(t-h_i(t))-y(t-h_{2,i})]^T(U_{1,i})[y(t-h_i(t))-y(t-h_{2,i})], \quad (42)
\end{aligned}$$

$$\begin{aligned}
&-(\tau_{2,i}-\tau_{1,i})\int_{t-\tau_{2,i}}^{t-\tau_{1,i}}\dot{y}^T(s)U_{2,i}\dot{y}(s)ds\\
&\leq [y(t-\tau_i(t))-y(t-\tau_{2,i})]^T(-U_{2,i})[y(t-\tau_i(t))-y(t-\tau_{2,i})]\\
&+[y(t-\tau_i(t))-y(t-\tau_{2,i})]^T(-U_{2,i})[y(t-\tau_i(t))-y(t-\tau_{2,i})]\\
&-\beta_\tau[y(t-\tau_{1,i})-y(t-\tau_i(t))]^T(U_{2,i})[y(t-\tau_{1,i})-y(t-\tau_i(t))]\\
&-(1-\beta_\tau)[y(t-\tau_i(t))-y(t-\tau_{2,i})]^T(U_{2,i})[y(t-\tau_i(t))-y(t-\tau_{2,i})]. \quad (43)
\end{aligned}$$

Furthermore, from the following zero equation

$$-P_i\dot{y}(t) + A_iP_iy(t) + D_{1,i}P_iy(t-h_i(t)) + D_{2,i}P_iy(t-\tau_i(t)) = 0,$$

we obtain

$$-2\dot{y}^TP_i\dot{y}(t) + 2\dot{y}^TA_iP_iy(t) + 2\dot{y}^TD_{1,i}P_iy(t-h_i(t)) + 2\dot{y}^TD_{2,i}P_iy(t-\tau_i(t)) = 0. \quad (44)$$

Hence, from (3.14),(3.16)-(3.26),(3.39)-(3.41), we get

$$\begin{aligned}
&\dot{V}_i(x_t) + \alpha V_i(x_t) \\
&\leq \xi_h^T(t)\Omega_{si}\xi_h(t) - \beta_h[y(t-h_{1,i}) - y(t-h_i(t))]^Te^{-\alpha h_{2,i}}(U_{1,i})[y(t-h_{1,i}) - y(t-h_i(t))] \\
&-(1-\beta_h)[y(t-h_i(t)) - y(t-h_{2,i})]^Te^{-\alpha h_{2,i}}(U_{1,i})[y(t-h_i(t)) - y(t-h_{2,i})] \\
&+\xi_\tau^T(t)\Phi_{si}\xi_\tau(t) - \beta_\tau[y(t-\tau_{1,i}) - y(t-\tau_i(t))]^Te^{-\alpha\tau_{2,i}}(U_{2,i})[y(t-\tau_{1,i}) - y(t-\tau_i(t))] \\
&-(1-\beta_\tau)[y(t-\tau_i(t)) - y(t-\tau_{2,i})]^Te^{-\alpha\tau_{2,i}}(U_{2,i})[y(t-\tau_i(t)) - y(t-\tau_{2,i})] \\
&= \xi_h^T(t)[(1-\beta_h)\Omega_{s1,i} + \beta_h\Omega_{s2,i}]\xi_h(t) \\
&+\xi_\tau^T(t)[(1-\beta_\tau)\Phi_{s1,i} + \beta_\tau\Phi_{s2,i}]\xi_\tau(t) \qquad (45)
\end{aligned}$$

where $\xi_h^T(t) = [y^T(t) \;\; \dot{y}^T(t) \;\; y^T(t-h_{1,i}) \;\; y^T(t-h_{2,i}) \;\; y^T(t-h_i(t))]$,
$\xi_\tau^T(t) = [y^T(t) \;\; \dot{y}^T(t) \;\; y^T(t-\tau_{1,i}) \;\; y^T(t-\tau_{2,i}) \;\; y^T(t-\tau_i(t))]$.
Since $0 \leq \beta_h \leq 1$, and $(1-\beta_h)\Omega_{s1,i} + \beta_h\Omega_{s2,i}$ is a convex combination of $\Omega_{s1,i}$ and $\Omega_{s2,i}$, $(1-\beta_h)\Omega_{s1,i} + \beta_h\Omega_{s2,i} < 0$ is equivalent to $\Omega_{s1,i} < 0$ and $\Omega_{s2,i} < 0$.
Similarly, we obtain that $(1-\beta_\tau)\Phi_{s1,i} + \beta_\tau\Phi_{s2,i} < 0$ is equivalent to $\Phi_{s1,i} < 0$ and $\Phi_{s2,i} < 0$.
Thus, we have

$$\dot{V}_i(x_t) + \alpha V_i(x_t) \leq 0. \quad (46)$$

From (3.43), when $t \in [t_{k-1}, t_k)$ we get

$$V_i(x_t) = V_{\sigma(t)}(x_t) \leq e^{-\alpha(t-t_{k-1})}V_{\sigma(t_{k-1})}(x_{t_{k-1}}). \quad (47)$$

For $i \in M_u$, we choose the Lyapunov-Krasovskii functional candidates as

$$V_i(x_t) = V_{1,i}(x_t) + ... + V_{11,i}(x_t), \quad (48)$$

where,
$V_{1,i}(x_t) = x^T(t)Y_ix(t)$,
$V_{2,i}(x_t) = \int_{t-h_{1,i}}^{t} x^T(s)e^{-\beta(s-t)}Y_iQ_{1,i}Y_ix(s)ds$,
$V_{3,i}(x_t) = \int_{t-h_{2,i}}^{t} x^T(s)e^{-\beta(s-t)}Y_iQ_{1,i}Y_ix(s)ds$,
$V_{4,i}(x_t) = h_{1,i}\int_{-h_{1,i}}^{0}\int_{t+s}^{t} \dot{x}^T(\theta)e^{-\beta(\theta-t)}Y_iR_{1,i}Y_i\dot{x}(\theta)d\theta ds$,

$$V_{5,i}(x_t) = h_{2,i}\int_{-h_{2,i}}^{0}\int_{t+s}^{t}\dot{x}^T(\theta)e^{\beta(\theta-t)}Y_iR_{1,i}Y_i\dot{x}(\theta)d\theta ds,$$
$$V_{6,i}(x_t) = (h_{2,i}-h_{1,i})\int_{-h_{2,i}}^{-h_{1,i}}\int_{t+s}^{t}\dot{x}^T(\theta)e^{-\beta(\theta-t)}Y_iU_{1,i}Y_i\dot{x}(\theta)d\theta ds,$$
$$V_{7,i}(x_t) = \int_{t-\tau_{1,i}}^{t}x^T(s)e^{-\beta(s-t)}Y_iQ_{2,i}Y_ix(s)ds,$$
$$V_{8,i}(x_t) = \int_{t-\tau_{2,i}}^{t}x^T(s)e^{-\beta(s-t)}Y_iQ_{2,i}Y_ix(s)ds,$$
$$V_{9,i}(x_t) = \tau_{1,i}\int_{-\tau_{1,i}}^{0}\int_{t+s}^{t}\dot{x}^T(\theta)e^{-\beta(\theta-t)}Y_iR_{2,i}Y_i\dot{x}(\theta)d\theta ds,$$
$$V_{10,i}(x_t) = \tau_{2,i}\int_{-\tau_{2,i}}^{0}\int_{t+s}^{t}\dot{x}^T(\theta)e^{-\beta(\theta-t)}Y_iR_{2,i}Y_i\dot{x}(\theta)d\theta ds,$$
$$V_{11,i}(x_t) = (\tau_{2,i}-\tau_{1,i})\int_{-\tau_{2,i}}^{-\tau_{1,i}}\int_{t+s}^{t}\dot{x}^T(\theta)e^{-\beta(\theta-t)}Y_iU_{2,i}Y_i\dot{x}(\theta)d\theta ds.$$

By the same previous argument, we obtain that

$$\dot{V}_i(x_t) - \beta V_i(x_t) \le 0. \tag{49}$$

From (3.46), when $t \in [t_{k-1}, t_k)$ we get

$$V_i(x_t) = V_{\sigma(t)}(x_t) \le e^{\beta(t-t_{k-1})}V_{\sigma(t_{k-1})}(x_{t_{k-1}}). \tag{50}$$

By using (3.10), (3.44) and (3.46), we have Thus

$$\begin{aligned} V_i(x_t) \;\; &\le \;\; \prod_{m=1}^{l(t)}\mu e^{\beta(t_m-t_{m-1})} \times \prod_{n=l(t)+1}^{N(t)-1}\mu e^{\alpha h}e^{-\alpha(t_n-t_{n-1})} \times V_{i_0}(x_{t_0})e^{-\alpha(t-t_{N(t)-1})} \\ &\le \prod_{m=1}^{l(t)}\mu e^{\beta(t_m-t_{m-1})} \times \prod_{n=l(t)+1}^{N(t)-1}\mu e^{\alpha h}e^{-\lambda^-(t_n-t_{n-1})} \\ &\times V_{i_0}(x_{t_0})e^{-\lambda^-(t-t_{N(t)-1})},\;\; t \ge t_0. \end{aligned} \tag{51}$$

Then Using (3.11) and (3.48), we have

$$V_i(x_t) \le \prod_{m=1}^{l(t)}\mu \times \prod_{n=l(t)+1}^{N(t)-1}\mu e^{\alpha h} \times V_{i_0}(x_{t_0})e^{-\alpha^*(t-t_0)},\;\; t \ge t_0.$$

By (3.12) and (3.13), we get

$$V_i(x_t) \le V_{i_0}(x_{t_0})e^{-(\alpha^*-\nu)(t-t_0)},\;\; t \ge t_0.$$

Thus, by (3.15), we have

$$\|x(t)\| \leq \sqrt{\frac{\lambda_1}{\lambda_2}}\|x_{t_0}\|_{c_h} e^{-\frac{1}{2}(\alpha^*-\nu)(t-t_0)},\ t \geq t_0.$$

Therefore, switched systems (3.1) is exponentially stable. □

3.2 Robust stability for switched systems with interval time-varying delay.

Theorem 3.2. *Let $\alpha > 0$ and $\beta > 0$ be the decay rate of the stable modes and the growth rate of the unstable modes, respectively. If there exists some positive numbers $\epsilon_{ji} (j = 1, 2, ..., 6, i \in M)$, some symmetric positive definite matrices $P_i, Q_{1,i}$, $Q_{2,i}, R_{1,i}, R_{2,i}, U_{1,i}, U_{2,i}$ such that the following conditions hold:*
$A1 - i$. For $i \in M_s$

$$\Theta_{s1,i} = \Theta_{si} - [0 \;\; 0 \;\; 0 \;\; -I \;\; I]^T \times e^{-\alpha h_{2,i}} U_{1,i} [0 \;\; 0 \;\; 0 \;\; -I \;\; I] < 0, \tag{52}$$

$$\Theta_{s2,i} = \Theta_{si} - [0 \;\; 0 \;\; I \;\; 0 \;\; -I]^T \times e^{-\alpha h_{2,i}} U_{1,i} [0 \;\; 0 \;\; I \;\; 0 \;\; -I] < 0, \tag{53}$$

$$\Theta_{s3,i} = \begin{bmatrix} \Theta_{s311,i} & P_i H_{1,i}^T & P_i H_{1,i}^T \\ * & -\frac{\epsilon_1}{2} I & 0 \\ * & * & -\frac{\epsilon_2}{2} I \end{bmatrix} < 0, \tag{54}$$

$$\Theta_{s4,i} = \begin{bmatrix} \Theta_{s411,i} & P_i H_{2,i}^T & P_i H_{2,i}^T \\ * & -\epsilon_3 I & 0 \\ * & * & -\epsilon_4 I \end{bmatrix} < 0, \tag{55}$$

$$\Xi_{s1,i} = \Xi_{si} - [0 \;\; 0 \;\; 0 \;\; -I \;\; I]^T \times e^{-\alpha \tau_{2,i}} U_{2,i} [0 \;\; 0 \;\; 0 \;\; -I \;\; I] < 0, \tag{56}$$

$$\Xi_{s2,i} = \Xi_{si} - [0 \;\; 0 \;\; I \;\; 0 \;\; -I]^T \times e^{-\alpha \tau_{2,i}} U_{2,i} [0 \;\; 0 \;\; I \;\; 0 \;\; -I] < 0, \tag{57}$$

$$\Xi_{s3,i} = \begin{bmatrix} \Xi_{s311,i} & P_i H_{1,i}^T & P_i H_{1,i}^T \\ * & -\frac{\epsilon_1}{2} I & 0 \\ * & * & -\frac{\epsilon_2}{2} I \end{bmatrix} < 0, \tag{58}$$

$$\Xi_{s4,i} = \begin{bmatrix} \Xi_{s411,i} & P_i H_{3,i}^T & P_i H_{3,i}^T \\ * & -\epsilon_5 I & 0 \\ * & * & -\epsilon_6 I \end{bmatrix} < 0, \tag{59}$$

$A1 - ii$. For $i \in M_u$

$$\Theta_{u1,i} = \Theta_{ui} - [0 \;\; 0 \;\; 0 \;\; -I \;\; I]^T \times e^{\beta h_{2,i}} U_{1,i} [0 \;\; 0 \;\; 0 \;\; -I \;\; I] < 0, \tag{60}$$

$$\Theta_{u2,i} = \Theta_{ui} - [0 \quad 0 \quad I \quad 0 \quad -I]^T \times e^{\beta h_{2,i}} U_{1,i} [0 \quad 0 \quad I \quad 0 \quad -I] < 0, \tag{61}$$

$$\Theta_{u3,i} = \begin{bmatrix} \Theta_{u311,i} & P_i H_{1,i}^T & P_i H_{1,i}^T \\ * & -\frac{\epsilon_1}{2} I & 0 \\ * & * & -\frac{\epsilon_2}{2} I \end{bmatrix} < 0, \tag{62}$$

$$\Theta_{u4,i} = \begin{bmatrix} \Theta_{u411,i} & P_i H_{2,i}^T & P_i H_{2,i}^T \\ * & -\epsilon_3 I & 0 \\ * & * & -\epsilon_4 I \end{bmatrix} < 0, \tag{63}$$

$$\Xi_{u1,i} = \Xi_{ui} - [0 \quad 0 \quad 0 \quad -I \quad I]^T \times e^{\beta \tau_{2,i}} U_{2,i} [0 \quad 0 \quad 0 \quad -I \quad I] < 0, \tag{64}$$

$$\Xi_{u2,i} = \Xi_{ui} - [0 \quad 0 \quad I \quad 0 \quad -I]^T \times e^{\beta \tau_{2,i}} U_{2,i} [0 \quad 0 \quad I \quad 0 \quad -I] < 0, \tag{65}$$

$$\Xi_{u3,i} = \begin{bmatrix} \Xi_{u311,i} & P_i H_{1,i}^T & P_i H_{1,i}^T \\ * & -\frac{\epsilon_1}{2} I & 0 \\ * & * & -\frac{\epsilon_2}{2} I \end{bmatrix} < 0, \tag{66}$$

$$\Xi_{u4,i} = \begin{bmatrix} \Xi_{u411,i} & P_i H_{3,i}^T & P_i H_{3,i}^T \\ * & -\epsilon_5 I & 0 \\ * & * & -\epsilon_6 I \end{bmatrix} < 0, \tag{67}$$

A2. For $\mu \geq 1$,

$$P_i \leq \mu P_j, Q_{r,i} \leq \mu Q_{r,j}, R_{r,i} \leq \mu R_{r,j}, U_{r,i} \leq \mu U_{r,j}, r = 1, 2, \forall i, j \in M. \tag{68}$$

A3. Let $T^+(t_0, t), T^-(t_0, t)$ *denote the total activation times of the unstable and stable modes over* (t_0, t)*, respectively. Assume that, for any* t_0*, the switching law guarantees that*

$$\inf_{t \geq 0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \geq \frac{\beta + \alpha^*}{\alpha + \alpha^*} \tag{69}$$

where $\alpha^* \in (0, \alpha)$*; furthermore, there exists* $0 < \nu < \alpha^*$ *such that*
(i) *for* $i \in M_s$

$$\ln \mu + \alpha h - \nu (t_k - t_{k-1}) \leq 0, k = r+1, r+2, ..., N. \tag{70}$$

(ii) *for* $i \in M_u$

$$\ln \mu - \nu (t_k - t_{k-1}) \leq 0, k = 1, 2, ..., r. \tag{71}$$

then switched systems (2.1) is exponentially stable,
where

$$\Theta_{si} = \begin{bmatrix} \Lambda_{s11,i} & \Lambda_{s12,i} & \Lambda_{s13,i} & \Lambda_{s14,i} & \Lambda_{s15,i} \\ * & \Lambda_{s22,i} & 0 & 0 & \Lambda_{s25,i} \\ * & * & \Lambda_{s33,i} & 0 & \Lambda_{s35,i} \\ * & * & * & \Lambda_{s44,i} & \Lambda_{s45,i} \\ * & * & * & * & \Lambda_{s55,i} \end{bmatrix},$$

$$\Xi_{si} = \begin{bmatrix} \Delta_{s11,i} & \Delta_{s12,i} & \Delta_{s13,i} & \Delta_{s14,i} & \Delta_{s15,i} \\ * & \Delta_{s22,i} & 0 & 0 & \Delta_{s25,i} \\ * & * & \Delta_{s33,i} & 0 & \Delta_{s35,i} \\ * & * & * & \Delta_{s44,i} & \Delta_{s45,i} \\ * & * & * & * & \Delta_{s55,i} \end{bmatrix},$$

$$\Theta_{ui} = \begin{bmatrix} \Lambda_{u11,i} & \Lambda_{u12,i} & \Lambda_{u13,i} & \Lambda_{u14,i} & \Lambda_{u15,i} \\ * & \Lambda_{u22,i} & 0 & 0 & \Lambda_{u25,i} \\ * & * & \Lambda_{u33,i} & 0 & \Lambda_{u35,i} \\ * & * & * & \Lambda_{u44,i} & \Lambda_{u45,i} \\ * & * & * & * & \Lambda_{u55,i} \end{bmatrix},$$

$$\Xi_{ui} = \begin{bmatrix} \Delta_{u11,i} & \Delta_{u12,i} & \Delta_{u13,i} & \Delta_{u14,i} & \Delta_{u15,i} \\ * & \Delta_{u22,i} & 0 & 0 & \Delta_{u25,i} \\ * & * & \Delta_{u33,i} & 0 & \Delta_{u35,i} \\ * & * & * & \Delta_{u44,i} & \Delta_{u45,i} \\ * & * & * & * & \Delta_{u55,i} \end{bmatrix},$$

$$\begin{aligned}
\Lambda_{s11,i} &= 0.5P_i[A_i + \alpha I]^T + 0.5[A_i + \alpha I]P_i + 2Q_{1,i} - 0.5(e^{-\alpha h_{2,i}} + e^{-\alpha h_{1,i}})R_{1,i} \\
&\quad + \tfrac{\epsilon_1}{2} E_1^T E_1 + \epsilon_3 E_2^T E_2 + \tfrac{\epsilon_7}{4} E_4^T E_4; \\
\Lambda_{s12,i} &= 0.5P_i A_i^T; \\
\Lambda_{s13,i} &= e^{-\alpha h_{1,i}} R_{1,i}; \\
\Lambda_{s14,i} &= e^{-\alpha h_{2,i}} R_{1,i}; \\
\Lambda_{s15,i} &= D_{1,i} P_i; \\
\Lambda_{s22,i} &= (h_{1,i}^2 + h_{2,i}^2) R_{1,i} + (h_{2,i} - h_{1,i})^2 U_{1,i} - 2P_i + \tfrac{\epsilon_2}{2} E_1^T E_1 + \epsilon_4 E_2^T E_2; \\
\Lambda_{s25,i} &= D_{1,i} P_i; \\
\Lambda_{s33,i} &= -e^{-\alpha h_{1,i}} Q_{1,i} - e^{-\alpha h_{1,i}} R_{1,i} - e^{-\alpha h_{2,i}} U_{1,i}; \\
\Lambda_{s35,i} &= e^{-\alpha h_{2,i}} U_{1,i}; \\
\Lambda_{s44,i} &= -e^{-\alpha h_{2,i}} Q_{1,i} - e^{-\alpha h_{2,i}} R_{1,i} - e^{-\alpha h_{2,i}} U_{1,i}; \\
\Lambda_{s45,i} &= e^{-\alpha h_{2,i}} U_{1,i}; \\
\Lambda_{s55,i} &= -1.5 e^{-\alpha h_{2,i}} U_{1,i};
\end{aligned}$$

$\Theta_{s311,i} = -0.5(e^{-\alpha h_{1,i}} + e^{-\alpha h_{2,i}})R_{1,i};$
$\Theta_{s411,i} = -0.5e^{-\alpha h_{2,i}}U_{1,i};$
$\Delta_{s11,i} = 0.5P_i[A_i + \alpha I]^T + 0.5[A_i + \alpha I]P_i + 2Q_{2,i} - 0.5(e^{-\alpha\tau_{2,i}} + e^{-\alpha\tau_{1,i}})R_{2,i}$
$\quad +\frac{\epsilon_1}{2}E_1^TE_1 + \epsilon_5E_3^TE_3;$
$\Delta_{s12,i} = 0.5P_iA_i^T;$
$\Delta_{s13,i} = e^{-\alpha\tau_{1,i}}R_{2,i};$
$\Delta_{s14,i} = e^{-\alpha\tau_{2,i}}R_{2,i};$
$\Delta_{s15,i} = D_{2,i}P_i;$
$\Delta_{s22,i} = (\tau_{1,i}^2 + \tau_{2,i}^2)R_{2,i} + (\tau_{2,i} - \tau_{1,i})^2U_{2,i} - 2P_i + \frac{\epsilon_2}{2}E_1^TE_1 + \epsilon_6E_3^TE_3;$
$\Delta_{s25,i} = D_{2,i}P_i;$
$\Delta_{s33,i} = -e^{-\alpha\tau_{1,i}}Q_{2,i} - e^{-\alpha\tau_{1,i}}R_{2,i} - e^{-\alpha\tau_{2,i}}U_{2,i};$
$\Delta_{s35,i} = e^{-\alpha\tau_{2,i}}U_{2,i};$
$\Lambda_{s44,i} = -e^{-\alpha\tau_{2,i}}Q_{2,i} - e^{-\alpha\tau_{2,i}}R_{2,i} - e^{-\alpha\tau_{2,i}}U_{2,i};$
$\Delta_{s45,i} = e^{-\alpha\tau_{2,i}}U_{2,i};$
$\Delta_{s55,i} = -1.5e^{-\alpha\tau_{2,i}}U_{2,i};$
$\Xi_{s311,i} = -0.5(e^{-\alpha\tau_{1,i}} + e^{-\alpha\tau_{2,i}})R_{2,i};$
$\Xi_{s411,i} = -0.5e^{-\alpha\tau_{2,i}}U_{2,i};$
$\Lambda_{u11,i} = 0.5P_i[A_i - \beta I]^T + 0.5[A_i - \beta I]P_i + 2Q_{1,i} - 0.5(e^{\beta h_{2,i}} + e^{\beta h_{1,i}})R_{1,i} + \frac{\epsilon_1}{2}E_1^TE_1$
$\quad +\epsilon_3E_2^TE_2 + \frac{\epsilon_7}{4}E_4^TE_4;$
$\Lambda_{u12,i} = 0.5P_iA_i^T;$
$\Lambda_{u13,i} = e^{\beta h_{1,i}}R_{1,i};$
$\Lambda_{u14,i} = e^{\beta h_{2,i}}R_{1,i};$
$\Lambda_{u15,i} = D_{1,i}P_i;$
$\Lambda_{u22,i} = (h_{1,i}^2 + h_{2,i}^2)R_{1,i} + (h_{2,i} - h_{1,i})^2U_{1,i} - 2P_i + \frac{\epsilon_2}{2}E_1^TE_1 + \epsilon_4E_2^TE_2;$
$\Lambda_{u25,i} = D_{1,i}P_i;$
$\Lambda_{u33,i} = -e^{\beta h_{1,i}}Q_{1,i} - e^{\beta h_{1,i}}R_{1,i} - e^{\beta h_{2,i}}U_{1,i};$
$\Lambda_{u35,i} = e^{\beta h_{2,i}}U_{1,i};$
$\Lambda_{u44,i} = -e^{\beta h_{2,i}}Q_{1,i} - e^{\beta h_{2,i}}R_{1,i} - e^{\beta h_{2,i}}U_{1,i};$
$\Lambda_{u45,i} = e^{\beta h_{2,i}}U_{1,i};$
$\Lambda_{u55,i} = -1.5e^{\beta h_{2,i}}U_{1,i};$
$\Theta_{u311,i} = -0.5(e^{\beta h_{1,i}} + e^{\beta h_{2,i}})R_{1,i};$
$\Theta_{u411,i} = -0.5e^{\beta h_{2,i}}U_{1,i};$
$\Delta_{u11,i} = 0.5P_i[A_i - \beta I]^T + 0.5[A_i - \beta I]P_i + 2Q_{2,i} - 0.5(e^{\beta\tau_{2,i}} + e^{\beta\tau_{1,i}})R_{2,i} + \frac{\epsilon_1}{2}E_1^TE_1$
$\quad +\epsilon_5E_3^TE_3;$
$\Delta_{u12,i} = 0.5P_iA_i^T;$
$\Delta_{u13,i} = e^{\beta\tau_{1,i}}R_{2,i};$
$\Delta_{u14,i} = e^{\beta\tau_{2,i}}R_{2,i};$
$\Delta_{u15,i} = D_{2,i}P_i;$
$\Delta_{u22,i} = (\tau_{1,i}^2 + \tau_{2,i}^2)R_{2,i} + (\tau_{2,i} - \tau_{1,i})^2U_{2,i} - 2P_i + \frac{\epsilon_2}{2}E_1^TE_1 + \epsilon_6E_3^TE_3;$
$\Delta_{u25,i} = D_{2,i}P_i;$
$\Delta_{u33,i} = -e^{\beta\tau_{1,i}}Q_{2,i} - e^{-\alpha\tau_{1,i}}R_{2,i} - e^{\beta\tau_{2,i}}U_{2,i};$
$\Delta_{u35,i} = e^{\beta\tau_{2,i}}U, i_{2,i};$

$\Lambda_{u44,i} = -e^{\beta\tau_{2,i}} Q_{2,i} - e^{-\alpha\tau_{2,i}} R_{2,i} - e^{\beta\tau_{2,i}} U_{2,i}$;
$\Delta_{u45,i} = e^{\beta\tau_{2,i}} U_{2,i}$;
$\Delta_{u55,i} = -1.5 e^{\beta\tau_{2,i}} U_{2,i}$;
$\Xi_{u311,i} = -0.5(e^{\beta\tau_{1,i}} + e^{\beta\tau_{2,i}}) R_{2,i}$;
$\Xi_{u411,i} = -0.5 e^{\beta\tau_{2,i}} U_{2,i}$.

Proof. By considering Lyapunov-krasovskii functional as in (3.14) and (3.45), we may proof this Theorem by using a similar argument as in the proof of Theorem 3.1. By replacing $A_i, D_{1,i}, D_{2,i}$ in (3.16) with $A_i + E_{1,i}F_{1,i}(t)H_{1,i}$, $D_{1,i} + E_{2,i}F_{2,i}(t)H_{2,i}$, $D_{2,i} + E_{3,i}F_{3,i}(t)H_{3,i}$, respectively. From applying Lemma 2.3, we get the following upper bounds for the uncertain terms of the systems (2.1),

$$2y^T(t)E_{1,i}F_{1,i}(t)H_{1,i}P_iy(t) \leq \epsilon_{1,i}y^T(t)E_{1,i}^TE_{1,i}y(t) + \epsilon_{1,i}^{-1}y^T(t)P_iH_{1,i}^TH_{1,i}P_iy(t),$$

$$2\dot{y}^T(t)E_{1,i}F_{1,i}(t)H_{1,i}P_iy(t) \leq \epsilon_{2,i}\dot{y}^T(t)E_{1,i}^TE_{1,i}\dot{y}(t) + \epsilon_{2,i}^{-1}y^T(t)P_iH_{1,i}^TH_{1,i}P_iy(t),$$

$$\begin{aligned} 2y^T(t)E_{2,i}F_{2,i}(t)H_{2,i}P_iy(t-h_i(t)) \leq\ & \epsilon_{3,i}y^T(t)E_{2,i}^TE_{2,i}y(t) \\ & +\epsilon_{3,i}^{-1}y^T(t-h_i(t))P_iH_{2,i}^TH_{2,i}P_iy(t-h_i(t)), \end{aligned}$$

$$\begin{aligned} 2\dot{y}^T(t)E_{2,i}F_{2,i}(t)H_{2,i}P_iy(t-h_i(t)) \leq\ & \epsilon_{4,i}\dot{y}^T(t)E_{2,i}^TE_{2,i}\dot{y}(t) \\ & +\epsilon_{4,i}^{-1}y^T(t-h_i(t))P_iH_{2,i}^TH_{2,i}P_iy(t-h_i(t)), \end{aligned}$$

$$\begin{aligned} 2y^T(t)E_{3,i}F_{3,i}(t)H_{3,i}P_iy(t-\tau_i(t)) \leq\ & \epsilon_{5,i}y^T(t)E_{3,i}^TE_{3,i}y(t) \\ & +\epsilon_{5,i}^{-1}y^T(t-\tau_i(t))P_iH_{3,i}^TH_{3,i}P_iy(t-\tau_i(t)), \end{aligned}$$

and

$$\begin{aligned} 2\dot{y}^T(t)E_{3,i}F_{3,i}(t)H_{3,i}P_iy(t-\tau_i(t)) \leq\ & \epsilon_{6,i}\dot{y}^T(t)E_{3,i}^TE_{3,i}\dot{y}(t) \\ & +\epsilon_{6,i}^{-1}y^T(t-\tau_i(t))P_iH_{3,i}^TH_{3,i}P_iy(t-\tau_i(t)). \end{aligned}$$

For $i \in M_s$, we obtain

$$\begin{aligned} \dot{V}_i(x_t) \leq\ & \xi_h^T(t)[(1-\beta_h)\Theta_{s1,i} + \beta_h\Theta_{s2,i}]\xi_h(t) + y^T(t)\Theta_{s31,i}y(t) \\ & +y^T(t-h_i(t))\Theta_{s41,i}y(t-h_i(t)) + \xi_\tau^T(t)[(1-\beta_\tau)\Xi_{s1,i} \\ & +\beta_\tau\Xi_{s2,i}]\xi_\tau(t) + y^T(t)\Xi_{s31,i}y(t) + y^T(t-\tau_i(t))\Xi_{s41,i}y(t-\tau_i(t)). \end{aligned}$$

For $i \in M_u$, we have

$$\begin{aligned} \dot{V}_i(x_t) \leq\ & \xi_h^T(t)[(1-\beta_h)\Theta_{u1,i} + \beta_h\Theta_{u2,i}]\xi_h(t) + y^T(t)\Theta_{u31,i}y(t) \\ & +y^T(t-h_i(t))\Theta_{u41,i}y(t-h_i(t)) + \xi_\tau^T(t)[(1-\beta_\tau)\Xi_{u1,i} + \beta_\tau\Xi_{u2,i}]\xi_\tau(t) \\ & +y^T(t)\Xi_{u31,i}y(t) + y^T(t-\tau_i(t))\Xi_{u41,i}y(t-\tau_i(t)), \end{aligned}$$

where

$$\begin{aligned}
\Theta_{s31,i} &= \Theta_{s311,i} + \epsilon_{1,i}^{-1} P_i H_{1,i}^T H_{1,i} P_i + \epsilon_{2,i}^{-1} P_i H_{1,i}^T H_{1,i} P_i \\
&\quad + \frac{\epsilon_{5,i}^{-1}}{2} B_i H_{3,i}^T H_{3,i} B_i^T + \frac{\epsilon_{6,i}^{-1}}{2} B_i H_{3,i}^T H_{3,i} B_i^T, \\
\Theta_{s41,i} &= \Theta_{s311,i} + \epsilon_{3,i}^{-1} P_i H_{2,i}^T H_{2,i} P_i + \epsilon_{4,i}^{-1} P_i H_{2,i}^T H_{2,i} P_i, \\
\Xi_{s31,i} &= \Xi_{s311,i} + \epsilon_{1,i}^{-1} P_i H_{1,i}^T H_{1,i} P_i + \epsilon_{2,i}^{-1} P_i H_{1,i}^T H_{1,i} P_i \\
&\quad + \frac{\epsilon_{5,i}^{-1}}{2} B_i H_{3,i}^T H_{3,i} B_i^T + \frac{\epsilon_{6,i}^{-1}}{2} B_i H_{3,i}^T H_{3,i} B_i^T, \\
\Xi_{s41,i} &= \Xi_{s311,i} + \epsilon_{3,i}^{-1} P_i H_{2,i}^T H_{2,i} P_i + \epsilon_{4,i}^{-1} P_i H_{2,i}^T H_{2,i} P_i,
\end{aligned}$$

$$\begin{aligned}
\Theta_{u31,i} &= \Theta_{u311,i} + \epsilon_{1,i}^{-1} P_i H_{1,i}^T H_{1,i} P_i + \epsilon_{2,i}^{-1} P_i H_{1,i}^T H_{1,i} P_i \\
&\quad + \frac{\epsilon_{5,i}^{-1}}{2} B_i H_{3,i}^T H_{3,i} B_i^T + \frac{\epsilon_{6,i}^{-1}}{2} B_i H_{3,i}^T H_{3,i} B_i^T, \\
\Theta_{u41,i} &= \Theta_{u311,i} + \epsilon_{3,i}^{-1} P_i H_{2,i}^T H_{2,i} P_i + \epsilon_{4,i}^{-1} P_i H_{2,i}^T H_{2,i} P_i, \\
\Xi_{u31,i} &= \Xi_{u311,i} + \epsilon_{1,i}^{-1} P_i H_{1,i}^T H_{1,i} P_i + \epsilon_{2,i}^{-1} P_i H_{1,i}^T H_{1,i} P_i \\
&\quad + \frac{\epsilon_{5,i}^{-1}}{2} B_i H_{3,i}^T H_{3,i} B_i^T + \frac{\epsilon_{6,i}^{-1}}{2} B_i H_{3,i}^T H_{3,i} B_i^T, \\
\Xi_{u41,i} &= \Xi_{u311,i} + \epsilon_{3,i}^{-1} P_i H_{2,i}^T H_{2,i} P_i + \epsilon_{4,i}^{-1} P_i H_{2,i}^T H_{2,i} P_i.
\end{aligned}$$

Applying Lemma 2.2, the LMIs $\Theta_{s31,i} < 0, \Theta_{s41,i} < 0, \Theta_{u31,i} < 0$ and $\Theta_{u41,i} < 0$ ($\Xi_{s31,i} < 0, \Xi_{s41,i} < 0, \Xi_{u31,i} < 0$ and $\Xi_{u41,i} < 0$) are equivalent to $\Theta_{s3,i} < 0, \Theta_{s4,i} < 0, \Theta_{u3,i} < 0$ and $\Theta_{u4,i} < 0$($\Xi_{s3,i} < 0, \Xi_{s4,i} < 0, \Xi_{u3,i} < 0$ and $\Xi_{u4,i} < 0$), respectively. By the same argument as in theorem 3.1, we may conclude that switched system (2.1) is exponentially stable. □

4 Numerical examples

In this section, we provide some example to illustrate the effectiveness of our results.

Example 4.1 Consider the following uncertain switched systems with switching multiple interval time-varying delays

$$\dot{x}(t) = \quad (A_i + \Delta A_i)x(t) + (D_{1,i} + \Delta D_{1,i})x(t - h_i(t)) + (D_{2,i} + \Delta D_{2,i})x(t - \tau_i(t)), i = 1, 2,$$

with

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, D_{1,1} = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, D_{2,1} = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.1 \end{bmatrix}, D_{1,2} = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, D_{2,2} = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix},$$

and

$$E_{j,i} = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.002 \end{bmatrix}, H_{1,i} = H_{2,i} = H_{3,i} = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.001 \end{bmatrix}, j = 1, ..., 3.$$

As an illustration, we choose $\alpha = 0.5, \beta = 0.5, \mu = 1.1, \alpha^* = 0.4, \nu = 0.2, \varepsilon_{l,i} = 1$ for $l = 1, 2, ...6, i = 1, 2,$ $h_1(t) = 0.1 + |\sin t|, h_2(t) = 0.1 + \sin^2 t, \tau_1(t) = 0.1 + |\cos t|, \tau_2(t) = 0.1 + cos^2 t,$ $F_{r,i}(t) = \begin{bmatrix} \sin t & 0 \\ 0 & \sin t \end{bmatrix}, r = 1, ..., 3$. In this case, we can take $h_1 = 0.1, h_2 = 1.1, \tau_1 = 0.1, \tau_2 = 1.1$. Then, by using the LMI control toolbox in Matlab, solutions of LMIs (3.49)-(3.64) are given by

$$P_1 = \begin{bmatrix} 3.8633 & -0.0013 \\ -0.0013 & 4.2309 \end{bmatrix}, Q_{11} = \begin{bmatrix} 1.0587 & -0.0181 \\ -0.0181 & 0.4464 \end{bmatrix}, Q_{21} = \begin{bmatrix} 1.0587 & -0.0181 \\ -0.0181 & 0.4464 \end{bmatrix},$$
$$R_{11} = \begin{bmatrix} 0.8698 & -0.0925 \\ -0.0925 & 0.7503 \end{bmatrix}, R_{21} = \begin{bmatrix} 0.8698 & -0.0925 \\ -0.0925 & 0.7503 \end{bmatrix}, U_{11} = \begin{bmatrix} 1.2384 & -0.0347 \\ -0.0347 & 1.4074 \end{bmatrix},$$
$$U_{21} = \begin{bmatrix} 1.2384 & -0.0347 \\ -0.0347 & 1.4074 \end{bmatrix}, P_2 = \begin{bmatrix} 3.8692 & -0.0009 \\ -0.0009 & 4.1717 \end{bmatrix}, Q_{12} = \begin{bmatrix} 0.2622 & -0.0002 \\ -0.0002 & 0.5450 \end{bmatrix},$$
$$Q_{22} = \begin{bmatrix} 0.2614 & -0.0002 \\ -0.0002 & 0.5404 \end{bmatrix}, R_{12} = \begin{bmatrix} 0.7220 & -0.0003 \\ -0.0003 & 0.9003 \end{bmatrix}, R_{22} = \begin{bmatrix} 0.6991 & -0.0002 \\ -0.0002 & 0.9036 \end{bmatrix},$$
$$U_{12} = \begin{bmatrix} 0.5501 & -0.0001 \\ -0.0001 & 0.5447 \end{bmatrix}, U_{22} = \begin{bmatrix} 0.5512 & -0.0001 \\ -0.0001 & 0.5435 \end{bmatrix}.$$

By computation, we obtain we obtain $T^- \geq 9T^+$. Given $T^+ = 1$ then $T^- \geq 9$.
By choosing initial condition as $\phi(t) = \begin{bmatrix} sin(t) \\ sin(t) \end{bmatrix}, t \in [-1.1, 0]$, the trajectories of solutions of the switched system and the trajectories of solutions of subsystems 1 and 2 for this example are shown in figure 1-3.

5 Conclusion

In this paper, we study the problem of exponential stability for a class of switched systems with switching multiple non-differentiable time-varying delays. Comparing with some existing results in the literature, the novelty of our results is twofold. Firstly, the state delay is time-varying in which the restriction on the derivative of the time-delay function is not required to design switching rule for the robust stability of the system and need not to use the free weighting matrix variables. Secondly, the obtained conditions for the exponential stability are delay-dependent and formulated in terms of the solution of standard LMIs which can be solved by various available algorithms. Numerical example is given to illustrate the effectiveness of the theoretical result.

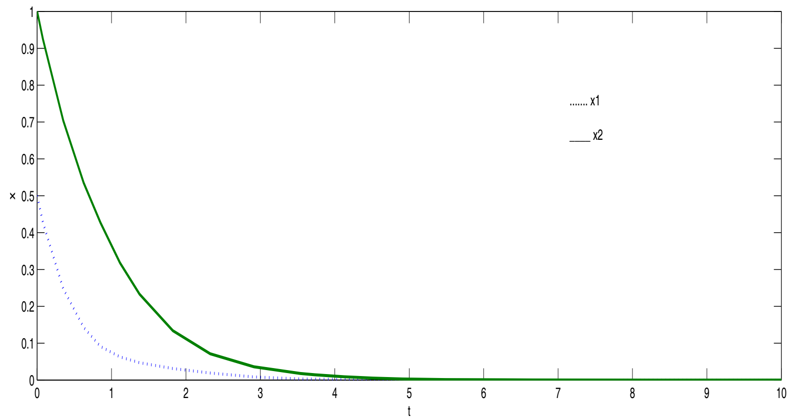

Figure 1: The trajectory of solution of subsystem $i = 1$.

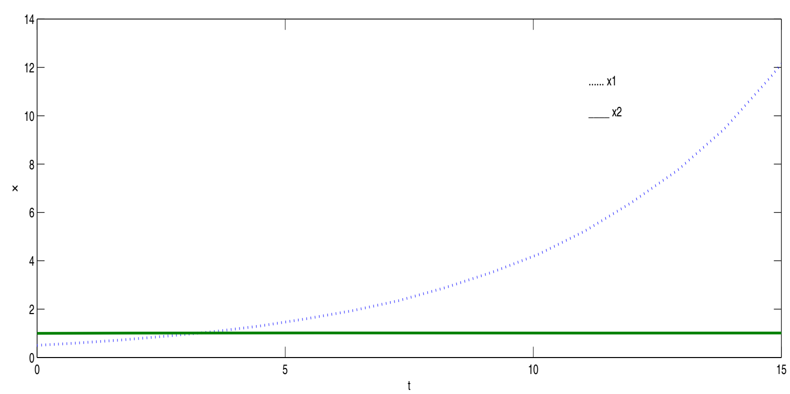

Figure 2: The trajectory of solution of subsystem $i = 2$.

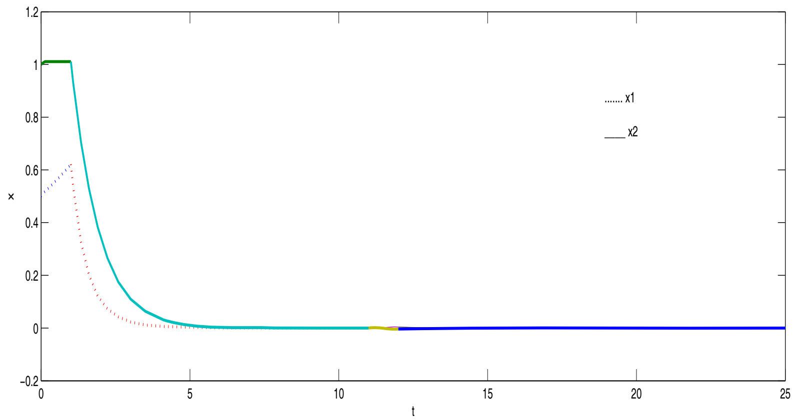

Figure 3: The trajectory of solution of the switched systems.

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